

The renormalized locally covariant Dirac field

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October 15, 2012

Abstract

The definition of the locally covariant Dirac field is adapted such that it may be charged under a gauge group and in the presence of generic gauge and Yukawa background fields. We construct renormalized Wick powers and time-ordered products. It is shown that the Wick powers may be defined such that the current and the stress-energy tensor are conserved, and the remaining ambiguity is characterized. We sketch a variant of the background field method that can be used to determine the renormalization group flow at the one loop level from the nontrivial scaling of Wick powers.

1 Introduction

The last one and a half decades saw an impressive revival of the theory of quantum fields on curved spacetimes. This was initiated by Radzikowski's discovery that Hadamard two-point functions can be equivalently characterized in terms of their wave front set [Rad96]. This lead Brunetti, Fredenhagen and Köhler to the formulation of the *microlocal spectrum condition* and the construction of Wick polynomials [BFK96]. Using a local renormalization scheme à la Epstein and Glaser and Steinmann's concept of the scaling degree, Brunetti and Fredenhagen were able to prove the perturbative renormalizability of the φ^4 model on generic spacetimes [BF00]. What was missing was some means to compare field theories defined on different spacetimes, or, put differently, to define one theory coherently on all spacetimes. This was provided by the *generally covariant locality principle* introduced by Brunetti, Fredenhagen and Verch [BFV03]. This principle is naturally formulated in categorical language: One starts with the category **Man** of globally hyperbolic manifolds, with causal isometric embeddings

as morphisms. A locally covariant theory is then a functor from **Man** to the category of (C^*) -algebras with injective homomorphisms as morphisms. The concept of a locally covariant theory was essential for the definition of covariant Wick powers and time-ordered products due to Hollands and Wald [HW01, HW02].

The framework was also crucial for the proof of the spin-statistics theorem on curved backgrounds [Ver01]. Examples of further applications are the discussion of quantum energy inequalities [FP06] and the renormalization group in curved spacetimes [HW03]. The framework was also used in the treatments of Yang–Mills gauge fields [Hol08], perturbative (classical) gravity [FR12], and the quantization of submanifold embeddings [BRZ12].

The locally covariant Dirac field was first considered by Sanders [San10b]. The crucial point in that work was the replacement of the category **Man** by the category **SpMan**, which also captures the spin structure. However, the spacetime and the spin-structure were the only allowed non-trivial backgrounds. Furthermore, only linear fields were incorporated, i.e., no Wick powers and time-ordered products. The latter problem was treated by Dappiaggi, Hack and Pinamonti, who provided a definition of Wick powers in order to be able to discuss backreaction effects through the semiclassical Einstein equation [DHP09]. But their proposal has some shortcomings, to be commented on below.

The aim of the present paper is to generalize and extend the framework of Sanders. The generalization consists in allowing for non-trivial gauge and Yukawa background fields. This is achieved by further extending the underlying category **SpMan** to the category **GSpMan**, which also includes the principal bundle corresponding to the gauge group, a gauge potential, and a scalar field (describing the Yukawa background). In particular, gauge transformations then correspond to morphisms of the category.

We extend Sanders’ work in that we also treat non-linear fields (Wick powers) and interactions (through time-ordered products). Building on the work of Rejzner [Rej11b] on fermionic fields on Minkowski space, we work in the framework of perturbative algebraic quantum field theory (pAQFT) [BDF09], i.e., by deformation quantization of a graded commutative algebra of functionals. The first step is to define the algebra of so-called *microcausal* functionals. The crucial point is to show that Hadamard two-point functions exist, a result that is a rather straightforward generalization of results of Fewster and Verch [FV02]. The next step is to define Wick powers. This is done via Hadamard parametrices, and the first task is to define what a covariant choice of a parametrix actually is. The next is then to show that parametrices exist. Our treatment requires less assumptions than the existing ones [SV01, DHP09], in that we allow for a coupling to non-trivial gauge and Yukawa backgrounds. Finally, we present a construction of time-ordered products, by a generalization of the work of Hollands and Wald for the scalar case [HW02].

We also provide some applications of the framework. We show that a conserved current operator can always be achieved and discuss the remaining renormalization freedom. This local and covariant definition of the current could also be useful for the study of backreaction effects in quantum electrodynamics on Minkowski space in the presence of an electromagnetic background field. Furthermore, we show that, provided the nontrivial background consists only of gravity and a constant mass, there is no algebraic obstruction to achieving a conserved stress-energy tensor, for any spacetime dimension. We also classify the remaining ambiguities, thereby proving a conjecture of [DHP09]. As another application, we sketch the determination of the renormalization group flow, at first order in \hbar , via a kind of background field method, solely on the basis of the scaling behavior of the parametrix, i.e., without calculating any loop integral.

The article is structured as follows: In the next section, we introduce the categorical setup, which now also includes a principal G -bundle and a background gauge connection and Yukawa field. We also introduce the classical algebra of functionals. In Section 3, the quantization of the algebra of functionals, via deformation quantization, is described. Also the construction of covariant Wick powers and time-ordered products is performed. The applications to current and stress-energy conservation and the renormalization group flow are described in Section 4.

1.1 Notation and Conventions

We are working on n -dimensional manifolds with signature $(-, +, \dots, +)$. For morphisms and equivalences of principal bundles, we use the following definition:

Definition 1.1. A *morphism* η between two principal G bundles P, P' over M and M' is a smooth map $\eta : P \rightarrow P'$ which is G -equivariant, i.e., $\eta(pg) = \eta(p)g$, and covers a smooth map $\chi : M \rightarrow M'$, i.e., $\pi' \circ \eta = \chi \circ \pi$. P and P' are *equivalent*, $P \simeq P'$, if η and χ are diffeomorphisms.

The Cartesian product of bundles E, F is denoted by $E \boxtimes F$, which is a bundle over the Cartesian product of the base spaces. Smooth sections of a bundle E with base space M are denoted by $\Gamma^\infty(M, E)$, and a subscript c denotes compactly supported sections. \dot{T}^*M denotes the cotangent bundle of M , with the zero removed. For a manifold M , $\mathcal{D}^k \subset M^k$ denotes the total diagonal,

$$\mathcal{D}^k = \{(x, \dots, x) \in M^k\}.$$

For a half-integer k , $[k]$ denotes the integer part. The symbol \doteq denotes a definition of the left hand side by the right hand side. Typically primed symbols, such as v' , stand for elements of a dual space (an exception is a primed coordinate x').

2 The categorial description

Before introducing the coupling to background fields, let us first review the structure introduced in [San10b]. The connected component of the Spin group is denoted by Spin_0 , cf. Appendix A for a definition. A *spin structure* SM over M is a principal Spin_0 bundle over M with a projection $\pi_S : SM \rightarrow FM$ to the frame bundle, which preserves the base point and intertwines the action of Spin_0 , i.e.,

$$\pi_S \circ S = \lambda(S) \circ \pi_S,$$

where $S \in \text{Spin}_0$ and λ is the covering map to the connected component Lor_0 of the Lorentz group. One defines the following category:

SpMan: The objects are spin structures SM whose base spaces M are oriented, time-oriented, globally hyperbolic manifolds. A morphism $\chi : SM \rightarrow SM'$ is a principal Spin_0 bundle morphism, covering an orientation, time-orientation and causality preserving isometric embedding $\psi : M \rightarrow M'$ such that $\pi'_S \circ \chi = \psi_* \circ \pi_S$.

In order to be able to functorially associate vector spaces and algebras to such spin manifolds, we also introduce the following categories.

Vec_(i): The objects are locally convex vector spaces. The morphisms are continuous linear (injective) maps.

Alg: The objects are topological $*$ -algebras. The morphisms are continuous injective $*$ -algebra homomorphisms.

As discussed in Appendix A, there is a standard (spinor) representation of Spin_0 on $\mathbb{C}^{[n/2]}$. The associated vector bundle DM induced by this representation is called the *standard* Dirac bundle in [San10b]. Its dual bundle is denoted by D^*M . We note that there are anti-linear conjugations¹

$$^+ : DM \rightarrow D^*M, \quad ^+ : D^*M \rightarrow DM,$$

fulfilling the usual properties, defined through

$$[p, z]^+ \doteq [p, z^+], \quad [p, z']^+ \doteq [p, z'^+],$$

where $p \in P$ and

$$z^+ \doteq -iz^*\gamma^0, \quad z'^+ \doteq -i\gamma^0 z'^*,$$

¹Of course there is also a charge conjugation. As our aim is to study arbitrary background gauge fields, where charge conjugation is not a symmetry, we do not discuss it here.

for $z \in \mathbb{C}^{[n/2]}$, $z' \in \mathbb{C}^{[n/2]*}$, cf. [San10b, Hac10] for details on the case $n = 4$.

We now want the Dirac field to be charged under a Lie group G in a representation ρ . Hence, we consider a principal G bundle P over M , and consider the direct product bundle² $SM + P$.³ On P , we consider a *connection*, i.e., a \mathfrak{g} valued horizontal 1-form A on P . We recall that the Levi-Civita connection induces a unique *spin connection* Ω on SM . By [KN63, Prop. II.6.3], there is then a unique connection on $SM + P$ such that the pushforward under the projection homomorphisms coincide with Ω and A . We also want to allow for couplings to a nonconstant Yukawa background field $m \in C^\infty(M)$. This leads us to consider the following category:

GSpMan: The objects are quadruples (SM, P, A, m) , where SM is a spin structure over an oriented, time-oriented globally hyperbolic manifold M , P a principal G bundle over M , A a connection on P , and $m \in C^\infty(M)$. A morphism $\chi : (SM, P, A, m) \rightarrow (SM', P', A', m')$ is given by (χ_{SM}, χ_P) , where $\chi_{SM(P)}$ is a principal $\text{Spin}_0(G)$ bundle morphism. χ_{SM} and χ_G cover the same orientation, time-orientation and causality preserving isometric embedding $\psi : M \rightarrow M'$ with $m = \psi^* m'$. Furthermore, $\pi' \circ \chi = \psi_* \circ \pi$ and $A = \chi_P^* A'$.

We note that a pair (χ_{SM}, χ_P) as above induces a principal $\text{Spin}_0 \times G$ bundle morphism $\chi : SM + P \rightarrow SM' + P'$ by $\chi(p, q) = (\chi_{SM}(p), \chi_P(q))$, where $p \in SM|_x$, $q \in P|_x$ for some $x \in M$. We also remark that taking $SM' = SM$, $P = P'$, $\chi_{SM} = \text{id}$, and, in a local trivialization, $\chi_P : (x, g) \mapsto (x, h(x)g)$ for some $h \in C^\infty(M, G)$ corresponds to a gauge transformation. Hence, gauge equivalence is built into the categorical framework.

Remark 2.1. The incorporation of background fields other than the gravitational one into the categorical framework can be found in earlier works, for example [BG11] (through the specification of a Green-hyperbolic operator) or [HW01] (though not formalized in the language of category theory). A unified treatment of gauge and general covariance can be found in [Hol08] (again not in the language of category theory). But, as explained below in Remark 3.4, our approach has a different notion of local covariance.

Given a representation ρ of G on a finite dimensional \mathbb{C} vector space V , we construct the vector bundle $D_\rho M$ associated to $SM + P$ via the representation $\rho_0 \otimes \rho$ on $\mathbb{C}^{[n/2]} \otimes V$. The corresponding dual bundle is denoted by $D_\rho^* M$, and the double spinor bundle by $D_\rho^\oplus M \doteq D_\rho M \oplus D_\rho^* M$.

²We refer to [KN63, p. 82] for a definition.

³Given the fact that we also might want to consider scalar fields charged under G , it seems reasonable to consider only direct products of SM and P and not general principal $\text{Spin} \times G$ bundles.

We define the vector spaces

$$\begin{aligned}\mathfrak{E}^{(*)}(SM, P) &\doteq \Gamma^\infty(M, D_\rho^{(*)}M), \\ \mathfrak{E}^\oplus(SM, P) &\doteq \Gamma^\infty(M, D_\rho^\oplus M).\end{aligned}$$

The assignments $(SM, P, A, m) \mapsto \mathfrak{E}^{(*)}(SM, P), \mathfrak{E}^\oplus(SM, P)$ are contravariant functors from **GSpMan** to **Vec**. Under \mathfrak{E} , the morphism χ is mapped to the pullback ξ^* of $\xi : D_\rho M \rightarrow D_\rho M'$, defined by $\xi([p, z]) = [\chi(p), z]$, $p \in SM + P$, $z \in \mathbb{C}^{[n/2]} \otimes V$, and analogously for \mathfrak{E}^* , \mathfrak{E}^\oplus . We also define the test section spaces

$$\begin{aligned}\mathfrak{D}^{(*)}(SM, P) &\doteq \Gamma_c^\infty(M, D_\rho M^{(*)}), \\ \mathfrak{D}^\oplus(SM, P) &\doteq \Gamma_c^\infty(M, D_\rho^\oplus M).\end{aligned}$$

These are covariant functors from **GSpMan** to **Vec_i**. A morphism χ is mapped to the push-forward ξ_* , where ξ is defined as above and ξ_* is extended from $\chi(M)$ to M' by the zero section. For later convenience, we also introduce

$$\mathfrak{T}_c(SM, P) \doteq \Gamma_c^\infty(M, \wedge(D_\rho^\oplus M \otimes T^\oplus M)), \quad (1)$$

where

$$T^\oplus M \doteq \bigoplus_k \text{Sym}^k TM,$$

\wedge denotes the exterior tensor product, and Sym^k the k th symmetric tensor product. This is also a covariant functor from **GSpMan** to **Vec_i**. Sometimes we need to be more specific, then \mathfrak{T}_c^{jA} denotes the subspace where the j th exterior power is taken, and $A \in \mathbb{N}_0^j$ counts the tensor power corresponding to $T^\oplus M$ in each of the factors. For example, $\mathfrak{T}_c^{10} = \mathfrak{D}^\oplus$.

As V is finite dimensional, $V \simeq \mathbb{C}^N$, there is an inner product on V . By averaging over G , we obtain a sesquilinear form $\langle \cdot, \cdot \rangle_V$ on V that is conserved under the action ρ . There is then a natural anti-linear map $^+$ from V to V^* , given by

$$v^+(w) \doteq \langle v, w \rangle_V.$$

Analogously, we may define $^+ : V^* \rightarrow V$. Thus, we may define the conjugation map $^+ : D_\rho M \rightarrow D_\rho^* M$ by

$$[p, z \otimes v]^+ \doteq [p, z^+ \otimes v^+], \quad p \in SM + P, z \in \mathbb{C}^{[n/2]}, v \in V,$$

and analogously for $^+ : D_\rho M^* \rightarrow D_\rho M$. This lifts to antilinear maps $\mathfrak{E}(SM, P) \rightarrow \mathfrak{E}^*(SM, P)$, $\mathfrak{E}^*(SM, P) \rightarrow \mathfrak{E}(SM, P)$, and hence to an anti-linear map $^+ : \mathfrak{E}^\oplus(SM, P) \rightarrow \mathfrak{E}^\oplus(SM, P)$. The pointwise pairing $D_\rho^* M|_x \times D_\rho M|_x \rightarrow \mathbb{C}$ defined by

$$\begin{aligned}\langle [p, z' \otimes v'], [p, z \otimes v] \rangle &\doteq z'(z)v'(v), \\ p \in SM + P, z \in \mathbb{C}^{[n/2]}, v \in V, z' \in \mathbb{C}^{[n/2]^*}, v' \in V^*,\end{aligned}$$

leads to a pairing $\mathfrak{E}^*(SM, P) \times \mathfrak{E}(SM, P) \rightarrow C^\infty(M)$, and to a pairing $\mathfrak{E}^\oplus(SM, P) \times \mathfrak{E}^\oplus(SM, P) \rightarrow C^\infty(M)$ defined by⁴

$$\langle (f, f'), (g, g') \rangle \doteq \langle g', f \rangle + \langle f', g \rangle, \quad f, g \in \mathfrak{E}(SM, P), f', g' \in \mathfrak{E}^*(SM, P). \quad (2)$$

2.1 The Dirac operator and its fundamental solutions

The connection on $SM + P$ induces the *exterior covariant derivative* d_A on $D_\rho M$, cf. [KN63, Sec. II.5]. This determines a covariant derivative ∇ on sections of $D_\rho M$, [KN63, Sec. III.1]. Analogously, there is a covariant derivative ∇^c on sections of $D_\rho^* M$. We may then define the Dirac operators D and D^* , which, in a local trivialization are given by

$$\begin{aligned} D &= -\gamma^\mu (\nabla_\mu^s - iA_\mu) + m = -\gamma^\mu \nabla_\mu + m, \\ D^* &= \gamma^\mu (\nabla_\mu^{s,c} + iA_\mu) + m = \gamma^\mu \nabla_\mu^c + m, \end{aligned}$$

where ∇^s and $\nabla^{s,c}$ are the spin connection, m is the smooth function in the objects of **GSpMan**, and A_μ is determined from the 1-form A in the objects of **GSpMan**. These operators intertwine the action of $\mathfrak{E}^{(*)}(\chi)$ and $\mathfrak{D}^{(*)}(\chi)$ for a morphism χ of **GSpMan**.

The squares of D and D^* are normally hyperbolic operators [BGP07], but they have a nontrivial first-order part. This is inconvenient for the computations we want to perform later on, so it is useful to introduce two auxiliary operators on spinors and co-spinors,

$$\tilde{D} \doteq -\gamma^\mu \nabla_\mu - m, \quad \tilde{D}^* \doteq \gamma^\mu \nabla_\mu^c - m.$$

We may then define the following operators on sections of $D_\rho M$ and $D_\rho^* M$:

$$\begin{aligned} P &\doteq -D\tilde{D} = -g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{R}{4} + \frac{i}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + m^2 - \gamma^\mu \partial_\mu m, \\ P_c &\doteq -D^* \tilde{D}^* = -g^{\mu\nu} \nabla_\mu^c \nabla_\nu^c + \frac{R}{4} - \frac{i}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + m^2 + \gamma^\mu \partial_\mu m. \end{aligned} \quad (3)$$

Here we used a local trivialization and defined the field strength as

$$F_{\mu\nu} \doteq \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

To the normally hyperbolic operator $P_{(c)}$ correspond unique retarded and advanced propagators

$$\Delta_{(c)}^{r/a} : \mathfrak{D}^{(*)}(SM, P) \rightarrow \mathfrak{E}^{(*)}(SM, P).$$

The corresponding propagators for $D^{(*)}$ are then defined as

$$S_{(c)}^{r/a} \doteq -\tilde{D}^{(*)} \Delta_{(c)}^{r/a}.$$

⁴Note that we are using a different convention than in [San10b] and [FV02], in that we are contracting the spinor with the cospinor and vice versa.

By construction, one then has $D^{(*)} \circ S_{(c)}^{r/a} = \text{id}$. A theorem by Dimock [Dim82] (see also [Müh11]), which is straightforwardly generalized to fields charged under a gauge group, implies that then also $S_{(c)}^{r/a} \circ D^{(*)} = \text{id}$ on $\mathfrak{D}^{(*)}(SM, P)$. Hence, $S_{(c)}^{r/a}$ are the unique retarded/advanced propagators for $D^{(*)}$, and the causal propagator is given by $S_{(c)} = S_{(c)}^r - S_{(c)}^a$. For the double spinor notation, we define⁵

$$D^\oplus \doteq D \oplus -D^*, \quad S^\oplus \doteq S \oplus -S_c.$$

As the $S_{(c)}^{r/a}$ are unique and $D^{(*)}$ intertwines $\mathfrak{E}^{(*)}$ and $\mathfrak{D}^{(*)}$, we have, for a morphism $\chi : (SM, P, A, m) \rightarrow (SM', P', A', m')$,

$$S_{(c)} = \mathfrak{E}(\chi) \circ S'_{(c)} \circ \mathfrak{D}(\chi).$$

We also note that $S_{(c)}$ fulfills

$$\int \langle f', Sf \rangle(x) d_g x = - \int \langle S_c f', f \rangle(x) d_g x = - \int \overline{\langle f^+, S f'^+ \rangle}(x) d_g x, \quad (4)$$

where $f \in \mathfrak{D}(SM, P)$, $f' \in \mathfrak{D}^*(SM, P)$, and $d_g x$ is the canonical volume form. The first equality can be shown as in the proof of Prop. 2.1 in [Dim82]. The second equality follows from $(Df)^+ = D^* f^+$ and the uniqueness of the retarded/advanced propagators. Finally, we remark that S^\oplus may also be seen as a distribution, $S^\oplus \in \Gamma_c^\infty(M^2, D_\rho^\oplus M \boxtimes D_\rho^\oplus M)'$, by

$$S^\oplus(u, v) \doteq \int \langle u, Sv \rangle(x) d_g x,$$

where we used the pairing (2). Analogously, $S \in \Gamma_c^\infty(M^2, D_\rho^* M \boxtimes D_\rho M)'$, and

$$S^\oplus((f, f'), (g, g')) = S(f', g) + S(g', f),$$

where we used (4).

2.2 Functionals

In the framework of pAQFT, one considers the algebra of functionals on the configuration space and deforms it (quantization). For fermionic fields, it was proposed in [Rej11b] to consider functionals on the space of antisymmetrized configurations, i.e., in the present setting, on

$$\wedge \mathfrak{E}^\oplus(SM, P) \doteq \bigoplus_{k=0}^{\infty} \wedge^k \mathfrak{E}^\oplus(SM, P),$$

⁵This is the Dirac operator obtained from the variation of the Dirac action, cf. [Rej11b]. In other works [DHP09, San10b], the double spinor Dirac operator is defined as $D \oplus D^*$.

with

$$\wedge^k \mathfrak{E}^\oplus(SM, P) \doteq \{B \in \Gamma^\infty(M^k, (D_\rho^\oplus M)^k) | B \text{ antisymmetric}\}.$$

This space is equipped with its natural topology (uniform convergence of all derivatives on compact subsets). For an element $B \in \wedge^k \mathfrak{E}^\oplus(SM, P)$, we denote by B_k its component in $\wedge^k \mathfrak{E}^\oplus(SM, P)$.

We now consider functionals on $\wedge^k \mathfrak{E}^\oplus(SM, P)$, i.e., linear maps from this space into the complex number. We denote by F_k the restriction of a functional F to $\wedge^k \mathfrak{E}^\oplus(SM, P)$. Then we define the grade by $|F_k| = k$. The *regular* functionals, $\mathfrak{F}_{\text{reg}}(SM, P)$, are those of the form

$$F_k(B) = \int \langle f_k, B_k \rangle(x_1, \dots, x_k) d_g x_1 \dots d_g x_k, \quad (5)$$

with $f_k \in \Gamma_c^\infty(M^k, D_\rho^\oplus M^k)$, f_k antisymmetric. We call f_k the *kernel* of F_k . Here we used the obvious generalization of the pairing (2). We can introduce an antisymmetric product \wedge on $\mathfrak{F}_{\text{reg}}(SM, P)$, by defining the kernel of the product $F \wedge G$ as

$$\begin{aligned} (f \wedge g)_k(x_1, \dots, x_k) \\ \doteq \sum_{l=0}^k \frac{1}{l!(k-l)!} \sum_{\pi \in S_k} (-1)^{|\pi|} f_l(x_{\pi(1)}, \dots, x_{\pi(l)}) f_{k-l}(x_{\pi(l+1)}, \dots, x_{\pi(k)}). \end{aligned}$$

An involution on $\mathfrak{F}_{\text{reg}}(SM, P)$ is defined as

$$F^*(B) \doteq \overline{F(B^+)}.$$

Finally, we equip $\mathfrak{F}_{\text{reg}}(SM, P)$ with the topology induced from the standard locally convex topology on $\Gamma_c^\infty(M^k, D_\rho^\oplus M^k)$ (uniform convergence of all derivatives on compact sets), the space of the kernels. The assignment $(SM, P, A, m) \mapsto \mathfrak{F}_{\text{reg}}(SM, P)$ is then a covariant functor from **GSpMan** to **Alg**.

The regular functionals do not allow for the description of local interactions or nonlinear observables, such as the stress-energy tensor. In order to cure this, one allows for more general kernels f_n , namely compactly supported distributions fulfilling the wave front set condition

$$\text{WF}(f_k) \cap (\bar{V}_+^k \cup \bar{V}_-^k) = \emptyset,$$

where \bar{V}_\pm is the closure of the dual of the forward/backward light cone. These are called the *microcausal* functionals. They also form an algebra $\mathfrak{F}(SM, P)$. It can be equipped with a topology such that it is a nuclear, locally convex vector space [BDF09, Rej11a]. \mathfrak{F} is then also a covariant functor from **GSpMan** to **Alg**.

The subspace $\mathfrak{F}_{\text{loc}}(SM, P)$ of $\mathfrak{F}(SM, P)$ in which the f_k 's are localized on the total diagonal \mathcal{D}^k and their wave front sets orthogonal to $T\mathcal{D}^k$,

$$\text{WF}(f_k) \perp T\mathcal{D}^k,$$

is the space of *local* functionals. It is a covariant functor from **GSpMan** to **Vec_i**.

We denote by $\mathfrak{F}_0(SM, P)$ the ideal of functionals that vanish on on-shell configurations, i.e., on configurations fulfilling $D^\oplus B = 0$, where D^\oplus acts on an arbitrary coordinate. We define the on-shell functionals as $\mathfrak{F}_S(SM, P) \doteq \mathfrak{F}(SM, P)/\mathfrak{F}_0(SM, P)$. This amounts to identifying two functionals if they agree on all on-shell configurations. Due to the functoriality of the Dirac operator, this is also a covariant functor from **GSpMan** to **Alg**.

3 Quantization

To prepare grounds for the deformation of the graded commutative algebra \mathfrak{F} in the spirit of deformation quantization [DF01], we first have to equip it with a Poisson structure by defining the Peierls bracket. To this avail, we first introduce functional derivatives [Rej11b]

$$F^{(1)}(B)(u) \doteq F(u \wedge B), \quad B \in \wedge \mathfrak{E}^\oplus(SM, P), u \in \mathfrak{E}^\oplus(SM, P).$$

Hence, $F^{(1)}(B)$ can be interpreted as a compactly supported distributional section of $D_\rho^\oplus M$. We denote its integral kernel by $F^{(1)}(B)(x)$. For $F \in \mathfrak{F}_{\text{reg}}$, this is even a smooth section. Given the fundamental solution S^\oplus , the Peierls bracket of two observables $F, G \in \mathfrak{F}_{\text{reg}}$ is defined as

$$[F, G] \doteq (-1)^{|F|+1} \int F^{(1)}(x) \wedge G^{(1)}(y) S^\oplus(x, y) d_g x d_g y.$$

Note that here and in the following, the contraction of $F^{(1)}$ and $G^{(1)}$ with S^\oplus has to be understood as the pairing defined in (2).

In deformation quantization, one aims at finding a product \star on the observables, fulfilling

$$F \star G = F \wedge G + \mathcal{O}(\hbar), \quad F \star G - (-1)^{|F||G|} G \star F = i\hbar[F, G] + \mathcal{O}(\hbar^2), \quad (6)$$

in the sense of formal power series in \hbar . This is straightforward for the regular functionals. We define the operator $\Gamma_{\frac{i}{2}S}^\otimes$ by

$$\Gamma_{\frac{i}{2}S}^\otimes(F \otimes G) \doteq (-1)^{|F|+1} \frac{i}{2} \int F^{(1)}(x) \otimes G^{(1)}(y) S^\oplus(x, y) d_g x d_g y,$$

and the \star product as

$$F \star G \doteq \wedge \exp(\hbar \Gamma_{\frac{i}{2}S}^\otimes) F \otimes G.$$

Here the wedge denotes the wedge product, $\wedge(F \otimes G) \doteq F \wedge G$. It is clear that (6) is fulfilled.

As S^\oplus is a bi-solution, \star is also well-defined on the regular on-shell functionals. As the fundamental solution is a local and covariant object, the assignment $(SM, P) \mapsto (\mathfrak{F}_{\text{reg}}(M)[[\hbar]], \star)$ is a covariant functor from **GSpMan** to **Alg**.

The extension to microcausal functionals proceeds via Hadamard two-point functions. These are defined as follows:

Definition 3.1. A *Hadamard two-point function* is a distributional section $\omega \in \Gamma_c^\infty(M^2, D_\rho^\oplus M^2)'$ fulfilling

$$\omega(D^\oplus u, v) = 0, \quad (7)$$

$$\omega(u, v) + \omega(v, u) = iS^\oplus(u, v), \quad (8)$$

$$\overline{\omega(u, v)} = \omega(v^+, u^+), \quad (9)$$

$$\text{WF}(\omega) \subset C_+, \quad (10)$$

where $u, v \in \Gamma_c^\infty(M, D_\rho^\oplus M)$ and

$$C_\pm = \{(x_1, x_2; k_1, -k_2) \in T^*M^2 \setminus \{0\} \mid (x_1; k_1) \sim (x_2; k_2), k_1 \in \bar{V}_{x_1}^\pm\}.$$

Here $(x_1; k_1) \sim (x_2; k_2)$ if there is a lightlike geodesic joining x_1 and x_2 to which k_1 and k_2 are co-parallel, and k_2 coincides with the parallel transport of k_1 along this curve. For $x_1 = x_2$, k_1, k_2 are lightlike and coinciding.

Assume for the moment that such distributions exist for all (SM, P) (this is shown later). Denote by $\omega_a(u, u') = \frac{1}{2}(\omega(u, u') - \omega(u', u))$ the anti-symmetric part and define a product \star_ω , equivalent to \star ,

$$F \star_\omega G \doteq \alpha_{\omega_a}^{-1} F \star \alpha_{\omega_a}^{-1} G, \quad (11)$$

by the equivalence map

$$\alpha_{\omega_a} \doteq \exp(\hbar \Gamma_{\omega_a}),$$

with

$$\Gamma_{\omega_a} F \doteq \int d_g x d_g y \omega_a(x, y) F^{(2)}(x, y).$$

By (8), the \star_ω product amounts to replacing $\frac{i}{2}S^\oplus$ by ω in the definition of \star . The condition (9) ensures that \star_ω is compatible with the conjugation. From (7) it follows that also \star_ω is well-defined on on-shell functionals. Furthermore, due to condition (10), \star_ω can be extended to the microcausal functionals $\mathfrak{F}(SM, P)$. To achieve a fully covariant construction, it is convenient to consider all possible ω 's at the same time. Hence, we define $\text{Had}(SM, P)$ to be the set of all Hadamard two-point functions. One then defines $\mathfrak{A}(SM, P)$ as the space of families

$$F = \{F_\omega\}_{\omega \in \text{Had}(SM, P)}, \quad F_\omega \in \mathfrak{F}(SM, P)[[\hbar]]$$

fulfilling

$$F_{\omega'} = \exp(\hbar \Gamma_{\omega'_a - \omega_a}) F_{\omega}. \quad (12)$$

In particular, an element F of $\mathfrak{A}(SM, P)$ is entirely specified by F_{ω} for a single $\omega \in \text{Had}(SM, P)$. We can then equip $\mathfrak{A}(SM, P)$ with the product

$$(F \star G)_{\omega} = F_{\omega} \star_{\omega} G_{\omega}.$$

Note that the assignment $M \mapsto (\mathfrak{A}(SM, P), \star)$ is a covariant functor from **GSpMan** to **Alg**, which maps a morphism χ to the morphism χ_* defined by

$$(\chi_* F)_{\omega} = \chi_*(F_{\omega|_{M \times M}}), \quad (13)$$

where on the r.h.s. χ_* is the morphism of $\mathfrak{F}[[\hbar]]$. Furthermore, we define the algebra $\mathfrak{A}_S(SM, P)$ of on-shell functionals analogously to $\mathfrak{F}_S(SM, P)$. The local elements $\mathfrak{A}_{\text{loc}}(SM, P)$ of $\mathfrak{A}(SM, P)$ are defined as those for which $F_{\omega} \in \mathfrak{F}_{\text{loc}}(SM, P)[[\hbar]]$ for one (and hence all) ω . Again, $\mathfrak{A}_{\text{loc}}$ is a covariant functor from **GSpMan** to **Vec**.⁶

It remains to show that Hadamard two-point functions exist. To this avail, we use the deformation argument of [FNW81]. First of all, we have the following proposition, whose proof can be found in Appendix B.

Proposition 3.2. *Let M be globally hyperbolic, (SM, π_S) a spin structure over M , P a principal G bundle over M with connection A , $m \in C^{\infty}(M)$, and Σ a smooth Cauchy surface of M . There exist M' , \tilde{M} globally hyperbolic and diffeomorphic to M with spin structures (SM', π'_S) and $(S\tilde{M}, \tilde{\pi}_S)$, P' , \tilde{P} principal G bundles over M' , \tilde{M} with $P \simeq P' \simeq \tilde{P}$, connection A' , \tilde{A} , $m' \in C^{\infty}(M')$, $\tilde{m} \in C^{\infty}(\tilde{M})$, and smooth Cauchy surfaces Σ' , $\tilde{\Sigma}' \subset M'$, $\tilde{\Sigma} \subset \tilde{M}$, $\Sigma' \cap \tilde{\Sigma}' = \emptyset$ such that*

1. Σ and Σ' are isometric and there are neighborhoods U , U' of Σ , Σ' such that U and U' are isometric and $m = i^* m'$ for this isometry. If i_P is the bundle isomorphism $i_P : P \rightarrow P'$ we have $A|_U = i_P^* A'|_U$. Furthermore, $(SM|_U, \pi_S) \simeq (SM'|_{U'}, \pi'_S)$.
2. \tilde{M} is ultrastatic, i.e., $\tilde{M} = \mathbb{R} \times \tilde{\Sigma}$ with metric $g = -dt^2 \otimes h$, where h is a Riemannian metric on $\tilde{\Sigma}$. Furthermore, $\tilde{m}(t, x) = m(x)$ in these coordinates. If $\{\tilde{U}_i\}_{i \in I}$ is a covering of $\tilde{\Sigma}$ by charts, then on each $U_i = \mathbb{R} \times \tilde{U}_i$ there is a section s_i of P such that the pull-back A_i of A along s_i is of the form

$$A_{i,0} = 0, \quad A_{i,a}(t, x) = A_{i,a}(x),$$

⁶In [DHP09], the Hadamard parametrix is used instead of Hadamard two-point functions. However, as discussed below, the parametrix is in general only defined in a neighborhood of the diagonal. Hence, the construction proposed in [DHP09] does not work in general, i.e., one does not obtain a covariant functor.

in the natural coordinates on U_i . Analogously, there is a section \tilde{s} of $S\tilde{M}$ such that the time component of the spin connection vanishes and the spatial one are time-independent.

3. $\tilde{\Sigma}$ and $\tilde{\Sigma}'$ are isometric and there are neighborhoods \tilde{U} , \tilde{U}' of $\tilde{\Sigma}$, $\tilde{\Sigma}'$ such that \tilde{U} and \tilde{U}' are isometric and $\tilde{m} = \tilde{i}^* m'$ for this isometry. If \tilde{i}_P is the bundle isomorphism $\tilde{i}_P : \tilde{P} \rightarrow \tilde{P}'$ we have $\tilde{A}|_{\tilde{U}} = \tilde{i}_P^* A'|_{\tilde{U}'}$. Furthermore, $(S\tilde{M}|_{\tilde{U}}, \tilde{\pi}) \simeq (SM'|_{\tilde{U}'}, \pi')$.

On the ultrastatic spacetime \tilde{M} and in the slicing $\tilde{M} \simeq \mathbb{R} \times \tilde{\Sigma}$, the Dirac equation may now be written as

$$i\partial_t \psi + K\psi = 0,$$

where K is, in a local trivialization, given by

$$K\psi \doteq i\gamma^0 \gamma^a (\partial_a - i\tilde{\sigma}_a - i\tilde{A}_a)\psi - i\gamma^0 m\psi,$$

where σ is the spin connection. From [BE81, Thm. 2.54] we conclude that $\tilde{\Sigma}$ is complete. Then it follows from [Che73, Thm. 2.2], that K is an essentially self-adjoint operator on $L^2(\tilde{\Sigma}, D_\rho \tilde{M}|_{\tilde{\Sigma}})$, where the scalar product is defined through the fiber-wise pairing

$$\langle z_1 \otimes v_1, z_2 \otimes v_2 \rangle = \bar{z}_1 \cdot z_2 \langle v_1, v_2 \rangle_V.$$

on $D_\rho \tilde{M}|_x \simeq \mathbb{C}^4 \otimes V$. Note that here we are not using spinor conjugation.⁷ In the following we denote the self-adjoint extension of K also by K . We can now proceed as in [Wro11] to obtain distributional sections $\omega^\pm \in \Gamma_c^\infty(\tilde{M}^2, D_\rho^* \tilde{M} \boxtimes D_\rho \tilde{M})'$. These are bisolutions fulfilling

$$\text{WF } \omega^\pm \subset C_\pm, \quad (14)$$

$$\omega^+ + \omega^- = i\tilde{S}, \quad (15)$$

where \tilde{S} is the commutator function on $(S\tilde{M}, \tilde{P})$. We can then define the distributional section $\omega \in \Gamma_c^\infty(\tilde{M}^2, D_\rho^\oplus \tilde{M} \boxtimes D_\rho^\oplus \tilde{M})'$ by

$$\begin{aligned} \omega(f', f) &\doteq \frac{1}{2} \left(\omega^+(f', f) + \overline{\omega^+(f^+, f'^+)} \right), \\ \omega(f, f') &\doteq \frac{1}{2} \left(\omega^-(f', f) + \overline{\omega^-(f^+, f'^+)} \right), \\ \omega(f, g) &\doteq 0, \\ \omega(f', g') &\doteq 0, \end{aligned}$$

where $f, g \in \mathfrak{D}(SM, P)$, $f', g' \in \mathfrak{D}^*(SM, P)$. Then (7) follows from ω^\pm being bi-solutions, (9) follows by definition, and (10) follows from (14). Condition

⁷This scalar product stems from the standard inner product $(f, g) = -i \int_\Sigma \langle f^+, \gamma^\mu g \rangle n_\mu$ for $f, g \in L^2(\Sigma, D_\rho M)$.

(8) is a consequence of (15). Hence, ω is a Hadamard two-point function on \tilde{M} .⁸

It remains to transport ω to M . By the isometry of a neighborhood \tilde{U} of $\tilde{\Sigma}$ and a neighborhood \tilde{U}' of $\tilde{\Sigma}'$, we can push-forward ω to a distribution on $\tilde{U}' \times \tilde{U}'$. Using the equation of motion, we extend it to the entire $M' \times M'$. By the isometry of neighborhoods U', U of Σ' and Σ , we may transfer it to M and again use the equation of motion there to extend it to $M \times M$. Due to the coincidence of Cauchy data, it is clear that the symmetric part still coincides with the fundamental solution. It remains to show that the Hadamard property is conserved under the extension procedure. By the propagation of singularity theorem, one only has to show that no elements $(x, \xi; y, 0)$ or $(x, 0; y, \eta)$ may appear in the wave front set. As the two-point function ω gives rise to a quasi-free state on the Cauchy data on $\tilde{\Sigma}$, one may use the calculus of Hilbert space valued distributions and argue as in [San10b, Sec. 4.2] to show that the wave front set may not contain such elements. We have thus proven:

Theorem 3.3. *There exist Hadamard two-point functions on each $\mathfrak{F}(SM, P)$.*

3.1 Fields

In the setting of local covariant field theories, fields are objects defined on all backgrounds simultaneously, in a coherent way [BFV03]. In the categorical language, this is encoded in requiring that they are natural transformations $\Phi : \mathfrak{T}_c \rightarrow \mathfrak{A}_{\text{loc}}$, where \mathfrak{T}_c was defined in (1). An example for a field are the linear fields

$$\psi_{(SM, P)}(u)_\omega(B) \doteq \int \langle u, B_1 \rangle(x) d_g x, \quad u \in \mathfrak{T}_c^{10}(SM, P) = \mathfrak{D}^\oplus(SM, P), \quad (16)$$

which are natural transformations $\mathfrak{T}_c^{10} \rightarrow \mathfrak{A}_{\text{loc}}$. We note that there is no dependence on ω on the r.h.s., as all the operators $\Gamma_{\omega'_a - \omega_a}$, cf. (12), vanish on this functional, as it is linear in the configuration. We also note that it fulfills

$$\begin{aligned} \psi(u)^* &= \psi(u^+) \\ \psi(u) \star \psi(v) + \psi(v) \star \psi(u) &= i\hbar S^\oplus(u, v). \end{aligned}$$

By choosing u to be a pure cospinor (spinor), one obtains the usual spinor (cospinor) fields, which, in an abuse of notation, will be denoted by ψ and ψ^+ in Section 4.

⁸An equivalent approach for the construction of a Hadamard two-point function on \tilde{M} would be to consider the CAR-algebra corresponding to the above Hilbert space (supplemented by co-spinorial sections) and using the projection on the positive spectrum of K to define a state [Ara70]. The corresponding two-point function fulfills the wave front condition, by [SV00].

Remark 3.4. The fields we consider are in general not gauge invariant, but gauge covariant, in the sense that we may integrate configurations with test sections (elements of \mathfrak{T}_c) that also transform under the gauge group action. In this respect we differ from the setting of [Hol08], where the “local and covariant functionals” are required to be gauge invariant.

In contrast, the definition of nonlinear fields (Wick powers), i.e., natural transformations $\mathfrak{T}_c^{jA} \rightarrow \mathfrak{A}_{\text{loc}}$ for $j > 1$, is not straightforward. The problem is to define them on all backgrounds and at the same time fulfill the relations (12) and (13). The crucial point is to find a trivializing distribution H which is covariantly assigned to each background and is such that $\omega - H$ is smooth for all Hadamard two-point functions. These are the parametrices, which we define as follows:

Definition 3.5. A *parametrix* H is a covariant assignment $(SM, P, A, m) \rightarrow H \in \Gamma_c^\infty(U, D_\rho^\oplus M \boxtimes D_\rho^\oplus M)'$, where U is a neighborhood of the diagonal of $M \times M$, such that (8), (9), (10) hold. Covariance here means that for $\chi : D_\rho^\oplus M \rightarrow D_\rho^\oplus M'$ the bundle morphism corresponding to a morphism $(SM, P, A, m) \rightarrow (SM', P', A', m')$ we have that $H - \chi^* H'$ is smooth on the common domain and vanishing at the diagonal, together with all the derivatives.

We note that the choice of the domain U is irrelevant, as for our purposes H only needs to be known in an arbitrarily small neighborhood of the diagonal. The requirement of covariance is crucial for the constructions presented below to be covariant. To our opinion, this aspect is not properly emphasized in [DHP09], at least not explicitly. A consequence of the definition is the following:

Proposition 3.6. *The difference $H - \omega$ is smooth on the domain of U for any Hadamard two-point function ω and any parametrix H .*

This is basically Lemma 2.9 of [San10a]. For convenience, we include a proof.

Proof. The distributional sections ω and H share the same symmetric part, i.e., $\omega_s - H_s = 0$, where $\omega_s(u, u') \doteq \frac{1}{2}(\omega(u, u') + \omega(u', u))$. We also know that $\text{WF}(\omega - H) \subset C_+$. Assume that $p \in C_+$ is contained in $\text{WF}(\omega - H)$. As the distribution $(u, u') \mapsto \omega(u', u)$ has wave front set contained in C_- , and analogously for H , it follows that p is also contained in $\text{WF}(\omega_s - H_s)$, as it can not be cancelled by symmetrization of the distribution. But $\text{WF}(\omega_s - H_s)$ is empty, so $\omega - H$ is smooth. \square

Remark 3.7. Since Hadamard two-point functions exist, as proven above, it follows that a parametrix is a bi-solution up to smooth terms. Alternatively, one may argue as in the Note Added in Proof in [Rad96].

With a parametrix H , we may associate to a local functional $F \in \mathfrak{F}_{\text{loc}}$ an element of $\mathfrak{A}_{\text{loc}}$ by

$$(F_H)_\omega \doteq \exp(\hbar \Gamma_{\omega-H})F. \quad (17)$$

This is well-defined as $H - \omega$ is smooth and the values of all its derivatives on the diagonal are unambiguous. As we only act on local functionals, the expression is well-defined even though H is only defined in a neighborhood of the diagonal. There is a canonical natural transformation $\Psi : \mathfrak{T}_c \rightarrow \mathfrak{F}_{\text{loc}}$, defined by

$$\Psi_{(SM,P)}(t)(B) \doteq \sum_{k=0}^{\infty} \int \langle t_{a_1 \dots a_k}^{\underline{\mu}_1 \dots \underline{\mu}_k}, \nabla_{(\underline{\mu}_1)}^{\oplus 1} \dots \nabla_{(\underline{\mu}_k)}^{\oplus k} B_k \rangle(x_1, \dots, x_k) d_g x_1 \dots d_g x_k, \quad (18)$$

where $\nabla_{(\underline{\mu}_i)}^i$ denotes the symmetrized covariant derivative on the i th coordinate, with $\nabla^{\oplus} \doteq \nabla \oplus \nabla^c$. Composing $\Psi|_{\mathfrak{T}_c^{jA}}$ with the map (17), we obtain fields, called the *Wick powers*. Hence, given a parametrix, a plethora of fields is available. In order to show that parametrices exist, we first define an auxiliary concept.

Definition 3.8. A *pre-parametrix* H^\pm is a covariant assignment (in the sense of Definition 3.5) $(SM, P, A, m) \rightarrow H^\pm \in \Gamma_c^\infty(U, D_\rho^* M \boxtimes D_\rho M)'$, where U is a neighborhood of the diagonal of $M \times M$, such that

$$H^+(x, y) - H^-(x, y) - iS(x, y) \in \Gamma^\infty(U, D_\rho M \boxtimes D_\rho^* M), \quad (19)$$

$$\text{WF}(H^\pm) \subset C_\pm. \quad (20)$$

Given a pre-parametrix H' , we define a parametrix H by setting

$$\begin{aligned} H(f', f) &= \frac{1}{2} \left(H^+(f', f) + \overline{H^+(f^+, f'^+)} \right) - \frac{1}{4} \left(r(f', f) + \overline{r(f^+, f'^+)} \right), \\ H(f, f') &= -\frac{1}{2} \left(H^-(f', f) + \overline{H^-(f^+, f'^+)} \right) - \frac{1}{4} \left(r(f', f) + \overline{r(f^+, f'^+)} \right), \\ H(f, f) &= 0, \\ H(f', g') &= 0, \end{aligned}$$

where $f, g \in \mathfrak{D}(SM, P)$, $f', g' \in \mathfrak{D}^*(SM, P)$, and r is the smooth remainder term in (19). We note that (4) ensures (8).

The existence of parametrices now follows from the following theorem:

Theorem 3.9. *On suitably chosen neighborhoods U of the total diagonal pre-parametrices exist.*

Proof. The operator P defined in (3) is normally hyperbolic [BGP07]. Thus, on each causal domain⁹ Ω , there are *Hadamard coefficients* $V_k \in \Gamma^\infty(\Omega \times \Omega, D_\rho M \boxtimes D_\rho^* M)$, recursively defined via the transport equation

$$\nabla_{\nabla \Gamma} V_k - \left(-\frac{1}{2} \nabla^\mu \partial_\mu \Gamma - n + 2k \right) V_k = 2k P V_{k-1},$$

⁹A causal domain is a geodesically convex domain which is globally hyperbolic.

with the initial condition $V_0(x, x) = \text{id}_{D_\rho M_x}$. Here all derivatives act on the first coordinate and $\Gamma(x, x')$ is the negative of the squared geodesic distance along the unique geodesic connecting x and x' . There are then retarded and advanced propagators $\Delta^{r/a}$ for P , which can be approximated, up to regularity Γ^k , by [BGP07]

$$\Delta_k^{r/a} \doteq \sum_{j=0}^{[\frac{n}{2}]-1+k} V_j R_\pm(2+2j) \quad (21)$$

in terms of the Riesz distributions R_\pm which are defined as the analytic continuation (in α) of

$$R_\pm(\alpha; x, x') \doteq \begin{cases} C(\alpha, n) \Gamma(x, x')^{\frac{\alpha-n}{2}} & \text{for } x' \in J_\mp(x), \\ 0 & \text{otherwise,} \end{cases}$$

where $J_\pm(x)$ denotes the causal future/past of x and

$$C(\alpha, n) \doteq \frac{2^{1-\alpha} \pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)! (\frac{\alpha-n}{2})!}.$$

Likewise, there are parametrices h^\pm for P , formally given by

$$h^\pm \doteq \frac{1}{2\pi} \sum_{j=0}^{\infty} V_j T_\pm(2j+2), \quad (22)$$

where $T_\pm(j)$ are certain distributions, which for $j \in \{0, 2, 4, \dots\}$ are defined by

$$\begin{aligned} T_\pm(j) &\doteq \lim_{\varepsilon \rightarrow +0} \pi C(j, n) (-\Gamma \mp i\varepsilon\theta_0 + \varepsilon^2)^{\frac{j-n}{2}} & \text{for } j < n, \\ T_\pm(j) &\doteq \lim_{\varepsilon \rightarrow +0} (-1)^{\frac{j}{2}-[\frac{n}{2}]} \pi C(j, n) (-\Gamma \mp i\varepsilon\theta_0 + \varepsilon^2)^{\frac{j-n}{2}} & \text{for } j > n, \end{aligned}$$

for odd n and

$$\begin{aligned} T_\pm(j) &\doteq \lim_{\varepsilon \rightarrow +0} C'(j, n) (-\Gamma \mp i\varepsilon\theta_0 + \varepsilon^2)^{\frac{j-n}{2}} & \text{for } j < n, \\ T_\pm(j) &\doteq \lim_{\varepsilon \rightarrow +0} C(j, n) \Gamma^{\frac{j-n}{2}} \log(-\Gamma \mp i\varepsilon\theta_0 + \varepsilon^2) / \Lambda^2 & \text{for } j \geq n, \end{aligned}$$

for even n . Here Λ is some arbitrary length scale, C is as above and

$$C'(j, n) \doteq -\frac{2^{1-j} \pi^{\frac{2-n}{2}} (\frac{n-j}{2}-1)!}{(\frac{j}{2}-1)!}.$$

We also used the notation

$$\theta_0(x, x') = t(x) - t(x'),$$

where t is some time function. It is crucial to note that, for $j \in \{0, 2, 4, \dots\}$,

$$T_+(j) - T_-(j) = 2\pi i(R_+(j) - R_-(j)), \quad (23)$$

$$\text{WF}(T_\pm(j)) \subset C_\pm. \quad (24)$$

The series (22) does in general not converge, so one has to introduce a cut-off: Choose a smooth function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ supported in $[-2, 2]$ and identically 1 in $[-1, 1]$. Then there is a sequence $\{\alpha_j\}$ of positive reals, converging to 0, such that

$$h^\pm(x, x') = \frac{1}{2\pi} \sum_{j=0}^{\infty} \chi(\Gamma(x, x')/\alpha_j) V_j(x, x') T_\pm(2j+2)(x, x')$$

converges, cf. [BGP07] for a proof for the case of the retarded/advanced propagator. This definition is unique up to smooth terms. Due to (23) and (24), we then have

$$h^+ - h^- - i\Delta \in \Gamma^\infty(\Omega \times \Omega, D_\rho M \boxtimes D_\rho^* M), \quad (25)$$

$$\text{WF } h^\pm \subset C_\pm, \quad (26)$$

where $\Delta \doteq \Delta^r - \Delta^a$ is the causal propagator for P . Let $\{\Omega_i\}$ be a covering of M with causal domains. Then $U \doteq \cup_i \Omega_i \times \Omega_i$ is a neighborhood of \mathcal{D}^2 . By choosing a partition of unity $\{\chi_i\}$ of U , subordinate to the covering $\{\Omega_i \times \Omega_i\}$, we may define $h^\pm \doteq \chi_i h_i^\pm$, which fulfills (25) and (26) on U .

As discussed in Section 2.1, the causal propagator for the Dirac operator is given by $S = -\tilde{D}\Delta$. Hence, we define $H^\pm \doteq -\tilde{D}h^\pm$. From the above, it is clear that this is indeed a pre-parametrix. \square

Remark 3.10. The distributions T_\pm involve a term of the form $\log \Gamma_\varepsilon/\Lambda^2$, where Λ is a length scale that has to be introduced to make the logarithm well-defined.¹⁰ The scale Λ is arbitrary, but has to be fixed to the same value on all backgrounds. The need for such a scale plays an important role in the discussion of the axioms for time-ordered products in the following subsection and of the scaling behavior in Section 4.3.

Remark 3.11. The parametrix, and hence the Wick powers, is not unique. One may always modify the parametrix by a smooth, locally and covariantly constructed function. In Section 4, we elaborate on this, and show that this freedom may be used to achieve a conserved current and stress-energy tensor. In the present setting, by modifying the parametrix, one modified Wick squares and all higher order powers. For the scalar field, Hollands and Wald also allowed for redefinitions of the Wick powers that only affect the

¹⁰In [DHP09] it is proposed to choose Λ proportional to the inverse mass m^{-1} . This only works if the mass is constant and non-zero. In particular, this prescription violates the smoothness condition introduced below (adapted such that m is required to be constant).

k th and higher order powers, for an arbitrary k [HW01]. To achieve this in the present setting, one would have to add to $\hbar\Gamma_{\omega-H}$ in (17) operators of the form

$$\hbar^{k/2}\Gamma_{H_k}^k F \doteq \hbar^{k/2} \int H_k(x_1, \dots, x_k) F^{(k)}(x_1, \dots, x_k) d_g x_1 \dots d_g x_k,$$

where H_k is smooth, locally and covariantly constructed, and defined in a neighborhood of \mathcal{D}^k . But as such redefinitions are not necessary for the fulfillment of current and stress-energy conservation, we do not pursue this issue further.

3.2 Time-ordered products

We now discuss the construction of renormalized time-ordered products in our setting. Let us start with the following definition:

Definition 3.12. The vector space \mathfrak{MT}_c^k of k -local test tensors is defined as the $\mathbb{Z}/2$ -graded k -fold tensor product of \mathfrak{T}_c , where the grade of $B \in \mathfrak{T}_c^{jA}$ is $|B| = j \bmod 2$. A typical element is denoted by $B_1 \hat{\otimes} \dots \hat{\otimes} B_k$, where the hat indicates the graded tensor product. The vector space \mathfrak{MT}_c of multilocal test tensors is defined as the direct sum of the \mathfrak{MT}_c^k .

We may now introduce the notion of multilocal fields.

Definition 3.13. A multilocal field is a natural transformation

$$\Phi : \mathfrak{MT}_c \rightarrow \mathfrak{A},$$

where, by concatenation with the forgetful functor, we interpret \mathfrak{A} as a functor between **GSpMan** and **Vec**.

Obviously, this is a generalization of the notion of fields as introduced in Section 3.1. Often we will want to be more specific, and denote Φ_k^{jA} the induced natural transformation

$$\Phi_k^{jA} : \mathfrak{T}_c^{j_1 A_1} \hat{\otimes} \dots \hat{\otimes} \mathfrak{T}_c^{j_k A_k} \rightarrow \mathfrak{A}.$$

Here \underline{j} and \underline{A} are the multiindices containing the j_i , A_i . We recall that j stands for the number of fields, and $A \in \mathbb{N}_0^j$ for the number of derivatives on the separate fields.

There are several further conditions on time-ordered products. In order to formulate these, we introduce the concept of scaling: Given a background (SM, P, A, m) , we obtain another background (SM', P', A', m') by rescaling, i.e., $g' = \lambda^{-2}g$, $m' = \lambda m$, $A' = \lambda A$. Then there is a $*$ -isomorphism $\sigma_\lambda : \mathfrak{A}(SM', P') \rightarrow \mathfrak{A}(SM, P)$, defined by

$$\sigma_\lambda(F)_\omega(B_k) \doteq \lambda^{-k \frac{n+1}{2}} F_{\omega(\lambda)}(B_k).$$

Here we identified the sections B_k in $\wedge^k \mathfrak{E}^\oplus(SM, P)$ and $\wedge^k \mathfrak{E}^\oplus(SM', P')$ and used

$$\omega^{(\lambda)}(u, v) \doteq \lambda^{-n-1} \omega(u, v).$$

We refer to [HW01, Lemma 4.2] for a proof in the scalar case. For a multilocal field Φ_k , one may define another multilocal field $S_\lambda \Phi_k$ by

$$(S_\lambda \Phi_k)_{(SM, P)}(t) \doteq \lambda^{nk} \sigma_\lambda(\Phi_{k(SM', P')}(\chi^* t)),$$

where $t \in \mathfrak{M}\mathfrak{T}_c^k$ and χ^* is the pullback to the scaled background. The *scaling dimension* of a field Φ_k^{jA} is defined as

$$d_{\Phi_k^{jA}} = \sum_{i=1}^k \left(\frac{n-1}{2} j_i + |A_i| \right).$$

The time-ordered products are now multilocal fields that fulfill further axioms. First of all, we require them to be well-defined as natural transformations

$$T_k : \underbrace{\mathfrak{F}_{\text{loc}} \hat{\otimes} \dots \hat{\otimes} \mathfrak{F}_{\text{loc}}}_{k \text{ times}} \rightarrow \mathfrak{A}, \quad (27)$$

obtained by using Ψ , cf. (18), to map the elements of \mathfrak{T}_c to $\mathfrak{F}_{\text{loc}}$. Again, $\hat{\otimes}$ denotes the $\mathbb{Z}/2$ -graded tensor product where the grading refers to the grade of $F \in \mathfrak{F}_{\text{loc}}$ modulo 2. Due to the integration, this induces relations between time-ordered products with different numbers of derivatives, called the Leibniz rule in [HW05] and the Action Ward Identity in [DF04]. In order to formulate it, we introduce a notation that will also be useful later on. The time-ordered product T_k^{jA} may be seen as an \mathfrak{A} -valued distributional section. Given a local trivialization, we write its integral kernel as

$$T^{\underline{\alpha}_1 \dots \underline{\alpha}_k}(x_1, \dots, x_k),$$

where the $\underline{\alpha}_i$ are multiindices consisting of tuples $(a_l, \underline{\mu}_l)_{l \in \{1, \dots, j_i\}}$, where the a_l are spinorial and gauge indices and the $\underline{\mu}_l$ spacetime multiindices with $|\underline{\mu}_l| = A_i(l)$. The Leibniz rule can then be formulated as

$$\nabla_i^\mu T^{\underline{\alpha}_1 \dots \underline{\alpha}_k}(x_1, \dots, x_k) = \sum_{l=1}^{j_i} T^{\underline{\alpha}_1 \dots (\underline{\alpha}_i + l\mu) \dots \underline{\alpha}_k}(x_1, \dots, x_k) + \dots,$$

where $\underline{\alpha} + l\mu$ means adding μ to the multiindex $\underline{\mu}_l$ inside $\underline{\alpha}$, and the dots stand for lower order term obtained by symmetrizing the derivatives.

There are a couple of further conditions:

Support: The support of $T_k(t)$, $t \in \mathfrak{M}\mathfrak{T}_c^k$ is contained in $\text{supp}_M t$, defined as

$$\text{supp}_M t = \{x | (x, x_2, \dots, x_k) \in \text{supp } t\}.$$

Causal factorization: Let $t \in \mathfrak{M}\mathfrak{T}_c^k$, $t' \in \mathfrak{M}\mathfrak{T}_c^l$ be multilocal test sections such that $\text{supp}_M t$ has no intersection with the past of $\text{supp}_M t'$. Then

$$T_{k+l}(t \hat{\otimes} t') = T_k(t) \star T_l(t').$$

Scaling: The time-ordered products T_k^{jA} scale *almost homogeneously*, i.e., there are natural numbers c_k^{jA} such that

$$\left(\lambda \frac{\partial}{\partial \lambda} - d_{T_k^{jA}} \right)^{c_k^{jA}} S_\lambda T_k^{jA} = 0. \quad (28)$$

Microlocal spectrum condition: Let ω be a quasi-free Hadamard state on $\mathfrak{A}(SM, P)$. Then the wave front set of the distributional section $\omega(T^{\alpha_1 \dots \alpha_k}(x_1, \dots, x_k))$ is contained in $C_T^k \subset T^*M^k$, defined through decorated graphs, cf. [BF00, HW02].

Smoothness: The time-ordered products depend smoothly on the background fields. Thus, let g_s, A_s, m_s depend smoothly on a parameter $s \in \mathbb{R}$. Let ω^s be a family of Hadamard states on $\mathfrak{A}(SM^{(s)}, P^{(s)})$, with smooth truncated n -point functions that depend smoothly on s . One then requires that

$$\begin{aligned} & \text{WF} \left(\omega^{(s)} \left(T_k^{(s)}(x_1, \dots, x_k) \right) \right) \\ & \subset \left\{ (s, \sigma; \{x_i, \xi_i\}) \in \dot{T}^*(\mathbb{R} \times M^k) \mid (\{x_i, \xi_i\}) \in C_T^{k, (s)} \right\}. \end{aligned}$$

Analyticity: In the case of an analytic spacetime, the Wick products depend analytically on the background fields. This is made precise by a condition analogous to the one for smoothness.

There are further conditions which are most easily stated for time-ordered products interpreted as maps (27). However, it is clear that these can be reformulated for time-ordered products interpreted as multilocal fields.

Expansion: The time ordered product commutes with functional differentiation, i.e.

$$T(F_1 \hat{\otimes} \dots \hat{\otimes} F_k)^{(1)}(x) = \sum_{i=1}^k (-1)^{\sum_{l=1}^{i-1} |F_l|} T(F_1 \hat{\otimes} \dots \hat{\otimes} F_i^{(1)}(x) \hat{\otimes} \dots \hat{\otimes} F_k). \quad (29)$$

Unitarity: We have

$$T(F_1 \hat{\otimes} \dots \hat{\otimes} F_k)^* = \sum_{I_1 \sqcup \dots \sqcup I_j} (-1)^{n+j+\Pi} T\left(\bigotimes_{i \in I_1} F_i^*\right) \star \dots \star T\left(\bigotimes_{i \in I_j} F_i^*\right),$$

where $I_1 \sqcup \dots \sqcup I_j$ denotes all partitions of $\{1, \dots, k\}$ into nonempty, pairwise disjoint subsets. Π denotes a combinatorial factor, depending on the grades of the F_i and the partition, which accounts for the reordering of the F_i on the right hand side.

Equation of motion: If ψ denotes the linear field (16), then

$$T(\psi(D^\oplus u) \hat{\otimes} F_1 \hat{\otimes} \dots \hat{\otimes} F_k) = i \langle T(\bigotimes_i F_i)^{(1)}, u \rangle + \psi(D^\oplus u) \star T(\bigotimes_i F_i). \quad (30)$$

The time-ordered products of order 1 are simply the Wick powers, as defined by (17). As noted above, cf. Remark 3.11, these are not unique.

The rationale behind the axiom of almost homogeneous scaling is the following: Because the classical theory has homogeneous scaling, one would like to impose this condition also for the quantum theory. However, as discussed in Remark 3.10, the parametrix contains a logarithmic term for n even, which necessitates the choice of a scale. This breaks homogeneous scaling, and almost homogeneous scaling is the minimal generalization of homogeneous scaling such that Wick products exist. Also the extension of distributions necessary to define time-ordered products typically breaks scale invariance.

Due to the axiom of causal factorization, time-ordered products can be defined recursively, by extension of distributional sections defined on $M^k \setminus \mathcal{D}^k$ to M^k [BF00]. The important point is to ensure locality and local Lorentz and gauge covariance in this extension, to preserve the functoriality. For the scalar field, this was performed in [HW02], see also [Hol08]. In the following, we only describe the changes to the argument that are necessary to accomodate charged spinors.

Due to the Leibniz rule, the distributional sections T_k^{jA} are not independent. The action of the derivation defines the subspace of the Leibniz dependent ones. As in [HW02], we may choose a complement of this subspace and only have to define the time-ordered products on a basis of this complement.

One considers a small enough neighborhood U of a point (x, \dots, x) on the diagonal \mathcal{D}^k , and expands a time-ordered product T_0 defined up to \mathcal{D}^k into Hadamard-ordered ones, i.e.,

$$\begin{aligned} T_0^{\alpha_1 \dots \alpha_k}(x_1, \dots, x_k)_\omega \\ = \sum_{\underline{\beta}_i \subset \alpha_i} c_{\underline{\alpha}\underline{\beta}} t_0^{\underline{\beta}_1 \dots \underline{\beta}_k}(x_1, \dots, x_k) : \Psi^{\alpha_1 \setminus \underline{\beta}_1}(x_1) \dots \Psi^{\alpha_k \setminus \underline{\beta}_k}(x_k) :_{\omega, H}, \end{aligned}$$

where the c 's are combinatorial constants, t_0 a distributional section, and

$$: \Psi^{\alpha_1}(x_1) \dots \Psi^{\alpha_k}(x_k) :_{\omega, H} \doteq \exp(\hbar \Gamma_{\omega-H}) \Psi^{\alpha_1}(x_1) \dots \Psi^{\alpha_k}(x_k).$$

Here $\Psi^{\underline{\alpha}}(x)$ denotes the integral kernel of the map Ψ , cf. (18), interpreted as an $\mathfrak{F}_{\text{loc}}$ -valued distributional section. The form of the above expansion follows from (29), cf. the discussion in [HW02] for the scalar case. Because of (13) and the requirement on the parametrix, the distributions t_0 are gauge invariant in the following sense: The difference

$$\begin{aligned} \rho(g(x_1))_{\underline{\beta}_1}^{\underline{\alpha}_1} \dots \rho(g(x_k))_{\underline{\beta}_k}^{\underline{\alpha}_k} t_0^{\underline{\beta}_1 \dots \underline{\beta}_k} [gA_\mu g^{-1} + g\partial_\mu g^{-1}](x_1, \dots, x_k) \\ - t_0^{\underline{\alpha}_1 \dots \underline{\alpha}_k} [A_\mu](x_1, \dots, x_k) \end{aligned} \quad (31)$$

is smooth and vanishes, with all its derivatives, at the diagonal.

To extend t_0 to all of U , one proceeds as follows: Fix the last coordinate to x and describe the coordinates x_1, \dots, x_{k-1} by Riemannian normal coordinates ξ_1, \dots, ξ_{k-1} w.r.t. x . It then suffices to extend the resulting distribution on $\mathbb{R}^{n(k-1)} \setminus \{0\}$ to the origin. To do this in a local way, one performs a scaling expansion of t_0 . In Riemannian normal coordinates and in a given trivialization, one defines the following family of metrics, masses, and gauge potentials:

$$g_{\mu\nu}^{(s)}(\xi) \doteq g_{\mu\nu}(s\xi), \quad m^{(s)}(\xi) \doteq sm(s\xi), \quad A_\mu^{(s)}(\xi) \doteq sA_\mu(s\xi).$$

Now one Taylor expands t_0 around $s = 0$, i.e.,

$$t_0 = \sum_{l=0}^p \frac{1}{l!} \tau_{0,l} + r_{0,p}$$

with

$$\begin{aligned} \tau_{0,l}(\cdot, x) &\doteq \frac{d^l}{ds^l} t_0[g^{(s)}, m^{(s)}, A^{(s)}](\cdot, x) \Big|_{s=0}, \\ r_{0,p}(\cdot, x) &\doteq \frac{1}{p!} \int_0^1 (1-s)^p \frac{d^p}{ds^p} t_0[g^{(s)}, m^{(s)}, A^{(s)}](\cdot, x) ds. \end{aligned}$$

By choosing p large enough, one obtains a distribution $r_{0,p}$ with a low enough scaling degree to have a unique extension that preserves the scaling degree [BF00]. Hence, it suffices to extend the τ_0 's. As shown in the following, these may be decomposed as

$$\tau_{0,l}^{\underline{\alpha}}(y, x) = \sum_{\underline{a}\underline{\mu}} C_{\underline{a}\underline{\mu}}(x) \exp_x^* u_{0,l}^{\underline{\alpha}\underline{a}\underline{\mu}}(y).$$

Here C is a Lorentz and gauge tensor of mass dimension l built from $g_{\mu\nu}$, and (covariant derivatives) of the curvature, the mass, and the field strength, all evaluated at x . The index \underline{a} is a gauge multiindex. The distributions $u_{0,l}$ are spinorial, Lorentz, and gauge tensors, which are Lorentz invariant,

$$u_{0,l}^{\underline{\alpha}\underline{a}\underline{\mu}}(\Lambda(S)\cdot) = S_{\underline{\beta}}^{\underline{\alpha}} \lambda(S)_{\underline{\nu}}^{\underline{\mu}} u_{0,l}^{\underline{\beta}\underline{a}\underline{\nu}}(\cdot), \quad (32)$$

where $S \in \text{Spin}_0$ and the action on the α indices is on the spinorial and the tensorial component. They are also gauge invariant in the following sense:

$$\rho(g)_{\underline{\beta}}^{\alpha} \rho(g)_{\underline{b}}^a u_{0,l}^{\underline{\beta b \mu}} = u_{0,l}^{\underline{\alpha a \mu}}. \quad (33)$$

Their scaling degree is $q-l$, where q is the scaling degree of t_0 at the diagonal.

To prove this, one proceeds as follows: One assumes that $g_{\mu\nu}$, m , and A_μ are polynomials in ξ . They are thus entirely determined by the value of their derivatives, i.e., the jet space, at the origin. As a jet space basis of A_μ , we may choose the following:

$$\partial_{(\mu_1} \dots \partial_{\mu_k} A_{\nu)}, \quad \nabla_{(\mu_1} \dots \nabla_{\mu_k} F_{\nu)\lambda}, \quad k \in \mathbb{N}_0.$$

If we now consider the infinitesimal version of (31), we see that t_0 may not depend on the derivatives of A , as otherwise derivatives of the gauge parameter would appear which do not have to vanish at the diagonal. Hence, we may compute the $\tau_{0,l}$ as follows:

$$\begin{aligned} \tau_{0,k}(y, x) &= \sum_{k=\sum_j (j l_j + (j+2)m_j + (j+1)p_j)} c_{lmp} \\ &\quad \frac{\partial^{\sum (l_j + m_j + p_j)}}{\prod_j \partial^{l_j} g_{\mu\nu, \sigma_1 \dots \sigma_j} \partial^{m_j} F_{\mu\nu; \sigma_1 \dots \sigma_j} \partial^{p_j} m_{, \sigma_1 \dots \sigma_j}} t_0(y, x)|_{g=\eta, m=0, A=0} \\ &\quad \times \prod_j (g_{\mu\nu, \sigma_1 \dots \sigma_j})^{l_j} (F_{\mu\nu; \sigma_1 \dots \sigma_j})^{m_j} (m_{, \sigma_1 \dots \sigma_j})^{p_j}. \end{aligned}$$

Here c_{lmp} is a combinatorial factor. The factor in the third line then gives the tensor C , whereas the factor in the second line gives the distributions $u_{0,k}$. The gauge invariance (33) is now a consequence of (31) and the fact that we evaluate at $A = 0$. Note in particular that the distributions $u_{0,k}$ do not depend on the background fields any more, so they are “universal”, and their extension to the origin defines a coherent extension of t_0 on all backgrounds simultaneously. For a discussion of how to extend in a way that preserves Lorentz invariance (32) and almost homogeneous scaling, we again refer to [HW02]. The preservation of the gauge symmetry is then straightforward if the gauge group is compact: For an extension u define

$$\tilde{u}^{\underline{\alpha a \mu}} = \int_G \rho(g)_{\underline{\beta}}^{\alpha} \rho(g)_{\underline{b}}^a u_{\underline{c}}^{\underline{\beta b \mu}} dg.$$

One then proceeds as in [HW02] to arrive at:¹¹

Proposition 3.14. *There exist covariant time-ordered products for any compact gauge group.*

¹¹The argument showing that it is possible to redefine the time-ordered products such that (30) holds can be found in [HW05].

4 Currents and conservation laws

As discussed in Remark 3.11, the definition of Wick powers, i.e., time-ordered products at first order, is not unique, but allows for some renormalization freedom. For example, to modify the definition of the Wick power $\Psi = \psi_A^+ \psi^B$, we can modify the parametrix as

$$H_A^B(x, y) \rightarrow H_A^B(x, y) + \delta H_A^B(s(x, y)),$$

where $s(x, y)$ is the point $\gamma_{x,y}(1/2)$, where $\gamma_{x,y} : [0, 1] : M \rightarrow M$ is the unique geodesic from x to y , and $\delta H(z)$ is a covariant tensor defined from the jet of the background fields at z . Of course δH has to have the correct scaling dimension, so for $n = 4$, we have possibilities like

$$\delta H = \alpha_0 m^3 + \alpha_1 m R + \alpha_2 i m \gamma^\mu \gamma^\nu F_{\mu\nu} + \alpha_4 \gamma^\mu \nabla_\mu R.$$

Such a redefinition of course also affects the Wick product $\Psi_\mu = \nabla_\mu \psi_A^+ \psi^B$. However, it can also be redefined independently, by

$$H_A^B(x, y) \rightarrow H_A^B(x, y) + \delta H_{\mu A}^B(s(x, y)) \partial_x^\mu s(x, y), \quad (34)$$

where δH_μ is a covariant tensor of mass dimension 4 (for $n = 4$). The ambiguity in the definition of Wick powers was first discussed by Hollands and Wald for the scalar case [HW01]. They also showed that it may be used to achieve a conserved stress-energy tensor for the scalar field in dimension $n > 2$ [HW05]. In the following, we perform an analogous analysis for the Dirac field.

4.1 Current conservation

We want to show that for the case of electrodynamics, i.e., $G = U(1)$ and ρ the fundamental representation on $V = \mathbb{C}$, the renormalization freedom of the Wick power $\psi_A^+ \psi^B$ can be used to achieve a conserved current

$$j^\mu = \text{tr } \psi^+ \gamma^\mu \psi.$$

Assume that $\nabla_\mu j^\mu$ is not weakly vanishing, i.e., not vanishing on on-shell configuration. As $\nabla_\mu j^\mu$ vanishes weakly at the classical level, the violation must be of $\mathcal{O}(\hbar)$ and a c-number. Let us write the violation as

$$\nabla_\mu j^\mu = \nabla_\mu Q^\mu.$$

As the construction was local and covariant, $\nabla_\mu Q^\mu$ must be a locally and covariantly constructed scalar. If also Q^μ is locally and covariantly constructed, we may simply set

$$\delta H = -2^{-[n/2]} \gamma^\mu Q_\mu,$$

to achieve a conserved current. However, this need not be the case. A counterexample for even n would be the Chern–Simons type form

$$Q^\mu = \varepsilon^{\mu\nu\lambda_1\ldots\lambda_{n-2}} A_\nu F_{\lambda_1\lambda_2} \cdots F_{\lambda_{n-3}\lambda_{n-2}}.$$

The problem is in fact a cohomological one: Is there an n -form T which is covariant and exact, $T = dQ$, such that the $n-1$ form Q can not be chosen to be covariant. The gravitational algebraic Poincaré lemma [BDK90, BBH95] states that all such T 's are polynomials in Casimirs built from the curvature two-forms F and $R^{\mu\nu} M_{\mu\nu}$. Here $M_{\mu\nu}$ are generators of the Lorentz group. It follows at once that there are no anomalies in odd dimensions. For $n = 4$, the only possible anomalies are $T = F \wedge F$, $T = R^{\mu\nu} \wedge R^{\lambda\rho} g_{\mu\lambda} g_{\nu\rho}$ and $T = R^{\mu\nu} \wedge R^{\lambda\rho} \varepsilon_{\mu\nu\lambda\rho}$. The first two are odd under the parity transformation $\varepsilon \rightarrow -\varepsilon$. As our model is not chiral, and thus independent of the orientation, such expressions may not occur. The latter is the Gauß–Bonnet–Chern term. As it depends neither on the mass nor on the gauge connection, we may use results that are valid for the case of uncharged Dirac fields coupled to a constant Yukawa field (a mass). For these, it follows from [DHP09, Prop A.1] that the coinciding point limits of $\text{tr } D_x H(x, y)$ and $\text{tr } D_y^* H(x, y)$ coincide, which shows that

$$\nabla_\mu j^\mu = \text{tr } D^* \psi^+ \psi - \text{tr } \psi^+ D \psi$$

vanishes. Hence, we have shown that for odd n and for $n = 4$, the current is conserved. It is to be expected that this holds for all n .

Let us discuss the remaining ambiguity. A redefinition leading to

$$j^\mu \rightarrow j^\mu + r^\mu$$

would require the existence of a local and covariant covector r^μ that is conserved. The only such covector is the external current J^μ responsible for the background field. It follows that j^μ is uniquely defined up to multiples J^μ . In particular it is unique in regions that are void of charges and currents. The fact that there is an ambiguity proportional to the external current was already discussed by Schwinger [Sch48], in a setting where the external potential was treated as a perturbation. This can be interpreted as a charge renormalization. Evaluation in a state (which amounts to computing a certain limit of the difference of the corresponding two-point function and the parametrix) then yields the expectation value of the current, which could be used to estimate back-reaction effects.

Remark 4.1. For the case of a flat background, the use of a local renormalization scheme based on the parametrix was already proposed by Marecki [Mar03, Sec. VI.7].¹² However, the discussion of the ambiguities given there is not completely satisfactory, as the need for a covariant prescription seems

¹²In the seminal work of Euler and Heisenberg [HE36], one finds the same approach of subtracting a (essentially unique) reference object from the two-point (which involves the

not to be fully taken into account. Other definitions of the renormalized current one finds in the literature usually rely on the existence of a ground state, i.e., they require an ultrastatic spacetime with time-independent background fields [DM75]. The usual method to compute back-reaction effects is via effective actions, cf. [DR85, Dun04] for an overview. In this approach, the dependence on the state seems obscure. We plan to address the issue of consequences of the local definition in a forthcoming joint work with M. Wrochna.

4.2 The stress-energy tensor

The renormalization freedom of Wick powers was used by Hollands and Wald to construct a conserved stress-energy tensor in the scalar case [HW05]. Here, we perform the analogous analysis for the case of charged Dirac fields.

The first thing to notice is that the stress-energy tensor is in general only conserved if all fields are on-shell. Unless we are given a Lagrangean for the Yukawa background field m , variation w.r.t. m leads to $\text{tr } \psi^+ \psi = 0$. We thus have two choices: Either we assume that background fields are absent, with the possible exception of a constant mass (which does not lead to problems with the stress-energy tensor). Or we assume that the background fields are equipped with some Lagrangean. But then the coupling to the Dirac fields should be treated perturbatively, as otherwise terms involving the Wick square $\psi^+ \psi$ would enter the equation of motion of the background fields. But this reduces us to the first case for the free theory.

Hence, let us consider a charged Dirac field with a possibly non-zero mass m in a vanishing gauge field background. The stress-energy tensor for this field is given by [FR04]

$$T_{\mu\nu} = \text{tr} \left[\frac{1}{2} (\nabla_{(\mu} \psi^+ \gamma_{\nu)} \psi - \psi^+ \gamma_{(\mu} \nabla_{\nu)} \psi) - \frac{1}{2} g_{\mu\nu} \left(\nabla_\lambda \psi^+ \gamma^\lambda \psi - \psi^+ \gamma^\lambda \nabla_\lambda \psi + 2m \psi^+ \psi \right) \right], \quad (35)$$

where the trace is over gauge and spinor indices. In terms of the Wick squares defined above, this may be written as

$$T_{\mu\nu} = \text{tr} \left[\frac{1}{2} (\gamma_\nu \Psi_\mu + \gamma_\mu \Psi_\nu) - \frac{1}{4} (\gamma_\mu \nabla_\nu \Psi + \gamma_\nu \nabla_\mu \Psi) - g_{\mu\nu} \left(\gamma^\lambda \Psi_\lambda - \frac{1}{2} \gamma^\lambda \nabla_\lambda \Psi + m \Psi \right) \right].$$

choice of a state). This idea goes back to Dirac [Dir34]. However, the reference object Euler and Heisenberg employ is not the parametrix. The difference is not only that there they use the mass to fix the scale Λ , but also the Hadamard coefficients V_k disagree. For example, the coinciding point limit of the analog of V_1 vanishes in [HE36], in contrast to the parametrix, cf. (40). This seems to stem from [Hei34], where, for some unknown reason, only terms at most linear in the γ -matrices are considered.

For its divergence and trace, we obtain

$$\begin{aligned}\nabla^\mu T_{\mu\nu} &= \text{tr} \left[\frac{1}{2} (\gamma_\nu \nabla^\mu \Psi_\mu + \gamma^\mu \nabla_\mu \Psi_\nu - 2\gamma^\mu \nabla_\nu \Psi_\mu) \right. \\ &\quad \left. - \frac{1}{4} (\gamma_\nu \nabla^\mu \nabla_\mu \Psi - \gamma^\mu \nabla_\nu \nabla_\mu \Psi + \gamma^\mu R_{\mu\nu} \Psi) - m \nabla_\nu \Psi \right], \\ g^{\mu\nu} T_{\mu\nu} &= \text{tr} \left[(1-n) \gamma^\mu \Psi_\mu - \frac{1}{2} (1-n) \gamma^\mu \nabla_\mu \Psi - nm \Psi \right].\end{aligned}$$

Here we used

$$[\nabla_\mu, \nabla_\nu] \Psi = \mathfrak{R}_{\mu\nu} \Psi - \Psi \mathfrak{R}_{\mu\nu},$$

where \mathfrak{R} is the spin curvature tensor, which fulfills

$$\mathfrak{R}_{ab} \gamma^b = -\gamma^b \mathfrak{R}_{ab} = \frac{1}{2} R_{ab} \gamma^b.$$

As for the divergence of the current, the divergence of the stress-energy tensor is a c-number modulo a weakly vanishing functional. This c-number is of the form $\nabla^\mu Q_{\mu\nu}$. In the following we will assume that, as in the case of the current, $Q^{\mu\nu}$ is locally and covariantly constructed. For $n = 4$ this follows from the results of [DHP09]. For the generic case, it was conjectured in [HW05] that this is the case for all parity preserving models¹³. To achieve a conserved stress-energy tensor, one may then use the redefinition (34) of the parametrix to modify

$$\Psi_\mu \rightarrow \Psi_\mu - d_V^{-1} 2^{-[n/2]} \left(\gamma^\nu Q_{\mu\nu} - \frac{1}{n-1} \gamma_\mu Q_\lambda^\lambda \right),$$

where d_V is the dimension of the gauge representation. Note that such a redefinition does not affect the current, so both current and stress-energy conservation can be achieved. Also note that there are no restrictions on the dimension n , in contrast to the scalar case [HW05], where one has $n - 2$ in the denominator, so that one can achieve conservation only for $n > 2$. Hence, we have proven:

Proposition 4.2. *There are no algebraic obstructions to achieving a conserved stress energy tensor, in arbitrary dimension n .*

If the above assumption is valid, this implies that the Wick powers may indeed be modified such that the stress-energy tensor is conserved in any dimension.

Remark 4.3. There is another prescription for obtaining a conserved stress-energy tensor, due to Moretti [Mor03]. There, one directly changes the stress-energy tensor by adding a Wick monomial that vanishes on-shell. In the scalar theory, one uses

$$T'_{ab} = T_{ab} + c g_{ab} \varphi P \varphi,$$

¹³There are counterexamples in parity violating theories, cf. [AGW84]. The fact that these are all possible purely gravitational anomalies [BDK90] suggests that this is indeed fulfilled.

where P is the wave operator. In the case of the Dirac field, this was adapted as [DHP09]

$$T'_{ab} = T_{ab} + cg_{ab}\psi^+ D\psi.$$

While the two methods give the same expectation values of the stress energy tensor, and thus are equivalent for the purpose of discussing the semi-classical Einstein equation, there are important conceptual differences. As noted in [HW05], it seems highly unlikely that Moretti's description can be generalized to the interacting case, in contrast to the method of Hollands and Wald. In particular, the redefinition of the Wick powers such that the stress energy tensor is conserved is the first step in constructing time-ordered products that fulfill the principle of perturbative agreement¹⁴ of [HW05]. Such a choice of time-ordered products will automatically ensure the conservation of the stress-energy tensor also in the interacting case. The fact that in the two-dimensional scalar case a conserved stress-energy tensor can not be achieved by a redefinition of Wick powers, whereas no such restriction exists for Moretti's description, further shows that the two methods are not equivalent.

Let us close this section by discussing the remaining renormalization freedom for $n = 4$. After achieving a conserved stress-energy tensor, the remaining freedom must preserve this conservation. Hence, it may only be modified as

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \beta_0 I_{\mu\nu} + \beta_1 J_{\mu\nu} + \beta_2 m^2 G_{\mu\nu} + \beta_3 m^4, \quad (36)$$

where $I_{\mu\nu}$ and $J_{\mu\nu}$ are the two linearly independent conserved curvature tensors of dimension 4 (obtained by variation w.r.t. $g^{\mu\nu}$ of R^2 and $R_{\mu\nu}R^{\mu\nu}$). Such changes may indeed be achieved, by the redefinition (again performed via the redefinition (34) of the parametrix)

$$\Psi_\mu \rightarrow \Psi_\mu + \frac{1}{4}\gamma^\nu \left(\delta T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\delta T^\lambda_\lambda \right).$$

Hence, one has the same renormalization ambiguities of the stress-energy tensor as for scalar fields, as conjectured in [DHP09]. In particular, these suffice to cancel the term $\square R$ in the trace anomaly [DHP09] (as $I_{\mu\nu}$ and $J_{\mu\nu}$ have trace proportional to $\square R$). However, let us note that if we treat the Yukawa background field completely perturbatively, then $m = 0$ and the last two terms in (36) are absent. These are replaced by one new ambiguity, namely the stress-energy tensor of the background Yukawa field, at zeroth order in perturbation theory. Similarly, one may add a multiple of the stress-energy tensor of the gauge background field.

¹⁴It states that the physics is independent of the choice of a background, i.e., of the split of the action into a free and an interacting part (provided the free part is at most quadratic in the fields).

4.3 Scaling behavior

We briefly comment on how a variant of the background field method can be used to determine the scaling behavior or renormalization group flow at $\mathcal{O}(\hbar)$. For even dimension n , one has a non-trivial scaling behavior of Wick powers, since, as discussed in Remark 3.10, the parametrix involves a logarithmic term, which necessitates the choice of a scale Λ . But due to local covariance, this choice must be done simultaneously on all backgrounds. Hence, for a Wick square Ψ , we in general have

$$S_\lambda \Psi = \lambda^{d_\Psi} \Psi + \hbar r \log \lambda, \quad (37)$$

where r is a local covariant object, and d_Ψ is the scaling dimension of Ψ . In order to interpret this result, consider the backgrounds (M, g, A, m) and $(M, \lambda^2 g, \lambda^{-1} A, \lambda^{-1} m)$. The choice of a definition of a Wick square Ψ should correspond to the design of a corresponding measurement apparatus. This apparatus involves a linear length L , which by definition is the same on all backgrounds. Now the conformal map $(M, g) \rightarrow (M, \lambda^2 g)$ maps the apparatus to one of length λL . Hence, comparing Ψ and $S_\lambda \Psi$ amounts to comparing two definitions of Ψ related by a different choice of a length scale. This is obviously in close analogy to the comparison of field theories defined at different renormalization scales, which is the idea underlying the Callan–Symanzik equation. The difference is that in the present setting, it already applies to Wick powers. We refer to [HW03, BDF09] for a deeper discussion of the connection of scaling to the usual notions of the renormalization group flow.

As noticed in the preceding subsection, ultimately the background fields should be determined dynamically, i.e., they should be given some Lagrangean, which, for the sake of simplicity, we assume to be free. The coupling to the Dirac fermions is now an interaction term. Hence, we split the Yukawa and the gauge field into a free and an interacting part:

$$m = m_0 + m_1, \quad A^\mu = A_0^\mu + A_1^\mu.$$

The fields m_1 and A_1^μ will be quantized. We can split the Lagrangean into a free part L_0 (involving m_1 and A_1^μ at most quadratically), and an interaction part, given by

$$L_1 = m_1 \psi^+ \psi + i A_{1\mu} \psi^+ \gamma^\mu \psi. \quad (38)$$

In L_0 , no coupling of m_1 or A_1 to the Dirac fermion is present, so in particular the parametrix will not contain couplings between these field. Hence, as the fields m_1 and $A_{1\mu}$ enter linearly in (38), the anomalous scaling of this expression is completely determined by that of the Wick squares $\psi^+ \psi$ and $\psi^+ \gamma^\mu \psi$ (where a trace is understood).

In four spacetime dimensions, the auxiliary Hadamard parametrix is formally given by

$$h_{\Lambda}^{\pm}(x, x') = \frac{1}{16\pi^2} \lim_{\varepsilon \rightarrow \pm 0} \left(4 \frac{V_0(x, x')}{\Gamma_{\varepsilon}(x, x')} + \log \frac{-\Gamma_{\varepsilon}(x, x')}{\Lambda^2} V(x, x') \right),$$

where

$$V = \sum_{k=0}^{\infty} \frac{1}{2^{2k}(k+1)!k!} \Gamma^k V_{k+1}. \quad (39)$$

Noting that the Hadamard parametrix is obtained by applying \tilde{D} , we see that in order to compute the scaling behavior of the above expressions, we have to know the coinciding point limit of V_1 up to the first order derivative. For these, we obtain, for the case of electrodynamics ($G = U(1)$ and the fundamental representation)

$$\begin{aligned} [V_1] &= -\frac{1}{12}R - \frac{i}{4}[\gamma^{\lambda}, \gamma^{\rho}]F_{\lambda\rho} - m^2 + \gamma^{\lambda}\partial_{\lambda}m, \\ [\nabla_{\mu}V_1] &= -\frac{1}{24}\nabla_{\mu}R - \frac{i}{8}[\gamma^{\lambda}, \gamma^{\rho}]\nabla_{\mu}F_{\lambda\rho} - m\partial_{\mu}m + \frac{1}{2}\gamma^{\lambda}\nabla_{\mu}\partial_{\lambda}m \\ &\quad - \frac{1}{6}\nabla^{\lambda}(\mathfrak{R}_{\mu\lambda} - iF_{\mu\lambda}), \end{aligned} \quad (40)$$

where \mathfrak{R} is the spin curvature and the square brackets denote the coinciding point limit. For simplicity, we used m, A instead of m_0, A_0 . The coefficient r of non-trivial scaling in (37) is now proportional to

$$r \sim 2m_1\nabla^{\lambda}\nabla_{\lambda}m_0 - \frac{1}{3}Rm_0m_1 - 4m_1m_0^3 + \frac{4}{3}A_1^{\mu}\nabla^{\lambda}F_{0,\lambda\mu}.$$

This is, up to total derivatives, the expansion to linear order in m_1, A_1^{μ} of

$$-\nabla^{\lambda}m\nabla_{\lambda}m - \frac{1}{6}Rm^2 - m^4 - \frac{1}{3}F^{\mu\nu}F_{\mu\nu},$$

which is the Lagrangean for a conformally coupled scalar m^4 theory and a Yang–Mills Lagrangean. Up to the purely gravitational terms (which we can not obtain here, unless we also split the metric), this coincides with the a_4 term of the bosonic part of the spectral action of Chamseddine and Connes [CC97]. In the present setting we obtain it from the fermionic part through scale transformations, on generic globally hyperbolic spacetimes (in contrast to the compact Riemannian spaces needed for the spectral action). We note that working on compact Riemannian spaces and using a cut-off, all terms of the bosonic part of the spectral action can be obtained by scale transformations [AKL11].

It would certainly be desirable to better understand this, i.e., to establish a proof that the above method can really be used to calculate the renormalization group flow of the interacting model at the one loop level. It seems that this would be a prerequisite for the fulfillment of the principle of perturbative agreement introduced by Hollands and Wald [HW05].

Acknowledgments

I would like to thank Dorothea Bahns, Thomas-Paul Hack, Harold Steinacker, and especially Michał Wrochna for helpful discussions and remarks. I am very grateful to Kartik Prabhu for communicating a mistake in an earlier version of the manuscript and for discussions on this point. A large part of this work was carried out at the Courant Research Centre “Higher Order Structures” at the University of Göttingen. This work was supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen and by the Austrian Science Fund (FWF) under the contract P24713.

A The Spin group

We recall some basic material on the Spin group, cf. [LM89] for more details. We denote by $\text{Cl}(n)$ the real Clifford algebra corresponding to the bilinear form with signature $(-, +, \dots, +)$ on \mathbb{R}^n , i.e., the algebra generated by the identity $\mathbb{1}$ and elements e^μ subject to

$$\{e^\mu, e^\nu\} = 2\eta^{\mu\nu}\mathbb{1}.$$

By defining the involution $e^{\mu*} = -e^\mu$, one obtains the complexified Clifford algebra $\text{Cl}^c(n)$. It is isomorphic to $\text{Mat}_{\mathbb{C}}(2^{\lfloor n/2 \rfloor})$ for even n and to $\text{Mat}_{\mathbb{C}}(2^{\lfloor n/2 \rfloor}) \oplus \text{Mat}_{\mathbb{C}}(2^{\lfloor n/2 \rfloor})$ for odd n . Restricting to the first summand for odd n , one obtains an irreducible representation of $\text{Cl}^c(n)$ on $\mathbb{C}^{\lfloor n/2 \rfloor}$, with inner product given by

$$(v, w) = -i\langle v, \gamma^0 w \rangle_{\mathbb{C}^{\lfloor n/2 \rfloor}},$$

where γ^μ is the image of e^μ in this representation.

One defines the $\text{Spin}(n)$ group as

$$\text{Spin}(n) \doteq \{s \in \text{Cl}(n) | s = u_1 \dots u_{2k}, u_i \in \mathbb{R}^n, u_i^2 = \pm 1\},$$

where we identified \mathbb{R}^n with a subspace of $\text{Cl}(n)$. Its connected component is denoted by $\text{Spin}_0(n)$. There is a canonical homomorphism to the connected component Lor_0 of the Lorentz group $SO(n-1, 1)$. For $n > 2$, this is a double covering, whereas for $n = 2$, both groups are isomorphic to \mathbb{R} . The restriction of the above irreducible representation of $\text{Cl}^c(n)$ to $\text{Spin}_0(n)$ is the *spinor representation*.

B Deformation of the background

Proof of Proposition 3.2. By [BS03], M is diffeomorphic to $\mathbb{R} \times \Sigma$, and we define $M', \tilde{M} = M$ as smooth manifolds. Similarly, we define $P', \tilde{P} = P$ and

$SM', \tilde{SM} = SM$ as smooth principal bundles. With the induced metric h , (Σ, h) is a Riemannian manifold, and there exist a Riemannian metric \tilde{h} , conformal to h , such that (Σ, \tilde{h}) is complete [NO61]. We define $\tilde{M} = \mathbb{R} \times \Sigma$ with the metric $\tilde{g} = -dt^2 \otimes \tilde{h}$. By [BE81, Thm. 2.54], it is globally hyperbolic. Now one proceeds as in [FNW81, Prop. C.1] to define a metric on M' that interpolates between Σ (at $t = 0$) and $\tilde{\Sigma}$ (at $t = -1$).

Regarding the spin structure, we note that for $n > 2$, spin structures are classified (up to equivalence) by assigning a sign to each nontrivial cycle, indicating whether in the covering of the frame bundle by the spin bundle one changes the sheet when following the cycle. This is purely topological, so by choosing the same assignment as for (SM, π_S) , we define the spin structures (SM', π'_S) and $(\tilde{SM}, \tilde{\pi}_S)$. For $n = 2$, Spin_0 and Lor_0 are isomorphic, so the spin structure is unique (up to equivalence). Extending a section s of $\tilde{SM}|_\Sigma$ by parallel transport along the time direction to all of \tilde{M} , one obtains the section \tilde{s} .

It remains to discuss the principal bundle P and its connection. Choose the t coordinate of $M = \mathbb{R} \times \Sigma$ such that the $t = 0$ slice is the Cauchy surface Σ . Choose an open covering $\{V_i\}_{i \in I}$ of Σ , such that each $V_i \subset \Sigma$ is topologically trivial. Then $\{U_i\}_{i \in I}$ with $U_i = \mathbb{R} \times V_i$ is an open covering of $\mathbb{R} \times \Sigma$. We want to show that there are local trivializations $\{U_i, \hat{\phi}_i\}_{i \in I}$ such that the corresponding transition functions $\hat{g}_{ij} : U_i \cap U_j \rightarrow G$ are independent of t . As U_i is topologically trivial, the bundle $P|_{U_i}$ admits a section, denoted by s_i . The corresponding transition functions are denoted by g_{ij} . Now define $\hat{g}_{ij}(t, x) = g_{ij}(0, x)$. This is also a set of transition functions, which can be used to define a bundle \hat{P} with local trivializations $\{U_i, \hat{\phi}_i\}_{i \in I}$ that have the \hat{g}_{ij} as transition functions [KN63, Prop. I.5.2]. But the cross sections \hat{s}_i corresponding to these trivializations must be related to the cross sections s_i as $\hat{s}_i = s_i \lambda_i$ for a smooth $\lambda_i : U_i \rightarrow G$. Then $\hat{g}_{ij} = \lambda_j^{-1} g_{ij} \lambda_i$, so the bundles are equivalent, by [Nab11, Lemma 4.3.3].

Now let A_i be the pull-back of the connection A along the section \hat{s}_i . In local coordinates, it is given as $A_{i,0}(t, x)dt + A_{i,a}(t, x)dx^a$. We may thus define the one-forms \tilde{A}_i as

$$\tilde{A}_i(t, x) = A_{i,a}(0, x)dx^a.$$

These \tilde{A}_i satisfy the consistency condition,

$$\tilde{A}_i(x) = \text{ad}(\hat{g}_{ji}^{-1}(x))\tilde{A}_j(x) + \hat{g}_{ji}^*(x)\Theta,$$

Θ being the Maurer–Cartan form on G , as they are fulfilled by the $A_i(0, x)$ and the \hat{g}_{ij} are independent of t . Similarly, we define

$$A'_i(t, x) = f(t)A_i(t, x) + (1 - f(t))\tilde{A}_i(t, x),$$

where $f \in C^\infty(\mathbb{R}, [0, 1])$ and $f(t) = 1$ for $t > -1/4$ and $f(t) = 0$ for $t < -3/4$. For the Yukawa fields \tilde{m}, m' , one proceeds in the obvious way. \square

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